THEORETICAL MODEL OF SCATTERING OF ELECTROMAGNETIC RADIATION IN MEDIA WITH PERIODICALLY CHANGING PERMITTIVITY

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1. Introduction

The interaction between sound and optical radiation is usually termed acoustooptic interaction. An acoustooptic effect as a typical example of scattering of electromagnetic waves in transparent medium (water, $LiNbO_3$, GaP, TeO_2 etc.) with periodically changing permittivity is produced by generating an ultrasonic wave in an optically transparent material. Possible configuration is shown in Fig. 1. When an acoustic wave propagates in an optically transparent medium, it produces a periodic modulation of the index of refraction (or relative permittivity) via the elasto-optical effect. This provides a moving phase grating which may scatter (or diffract?) portions of an incident optical radiation into one or more directions. This phenomenon, known as the acoustooptic (AO) scattering, has led to a variety of optical devices that perform spatial, temporal, and spectral modulations of optical radiation. These devices have been used in optical systems for light-beam control and optical signal processing applications.

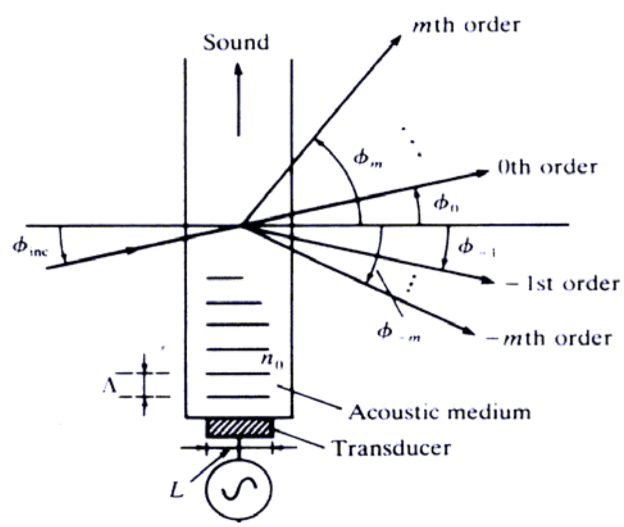

Fig. 1 *Basic acoustoptic device (reproduced from* [1] *)*

The acoustooptic effect is usually explained as a collision of photons and phonons [1]. For these quasiparticles to have a well defined momenta and energies, we must assume, that we have interaction of monochromatic plane waves of optical radiation and sound. The width of the transducer L must be assumed to be sufficiently wide in order to produce plane wave fronts at a single frequency. In the process of collisions two conservations laws have to be obeyed, namely, the conservation of energy and momentum. If $L \gg \Lambda^2/\lambda$ (Λ is the sound wavelength, λ is the optical radiation wavelength in the acoustic medium), only scaterring angles Φ_0 and Φ_{-1} are observed. In this case the acoustooptic device operates in the so called *Bragg regime* since this is similar to what happens in Bragg scattering of X-rays or electrons on a crystal lattice. The „grating constant“ in this case is equal to sound wavelength and the required angle of incidence Φ_{inc} is given by well-known Bragg condition $2\Lambda\sin\Phi_{inc} = \lambda$ [1, page 557].

In the case where L is sufficiently short, we have a second form of a scattering, called Raman-Nath (or Debye-Sears) scattering. The condition $L \ll \Lambda^2/\lambda$ defines therefore the *Raman-Nath scattering regime* [2]. In this regime many scattering angles may exist because various directions of plane waves of sound are provided from a small-aperture transducer.

In this work a possible semi-classical model of the intensity of scattered optical radiation in the Raman-Nath regime is proposed and briefly presented.

2. Theory

The acoustoptic device shown in Fig. 1 may be thought to act as a thin phase grating with an effective grating line separation equal to the wavelength Λ of the sound in acoustic and transparent medium. It is well known that a phase grating split incident optical radiation into various orders. The directions of the scattered (or diffracted) radiation inside the sound cell are governed by the grating equation [1]

$$\sin\Phi_m = \sin\Phi_{inc} + m\frac{\lambda}{\Lambda} = \sin\Phi_{inc} + m\frac{k}{K}; \quad m = 0; \pm 1, \pm 2, ... \tag{1}$$

Hence, the angle between neighboring orders in Fig. 1 is equal to λ/Λ inside the cell. (Outside the cell this angles are increased through refraction; k and K are propagation constants of the radiation and sound waves inside the cell.) The energy conservation law leads to the equation [1]

$$\omega_m = \omega \pm m\Omega \tag{2}$$

with ω_m being the frequency of the m order of scattered optical radiation, ω is the frequency of incident radiation and Ω is the frequency of sound wave (sound velocity about 1-7 km/s). The changes of the frequencies are due to multiple photon-phonon interactions (connected with emission or absorption of phonons). We will assume that x axis is in the sound direction and z axis lies in the direction from left to right in Fig. 1. The next analysis is in some aspects similar to work [3]. We assume an isotropic inhomogeneous non-magnetic non-conductive medium without birefringence.

The interaction between the electromagnetic field $\boldsymbol{E}(x, z, t)$ (polarized along the y axis) and sound field $S(z, x, t)$ can be described by Maxwell´s equations. The time varying permittivity is written as

$$\varepsilon(x,z,t) = \varepsilon + \varepsilon_1(x,z,t); \quad \varepsilon_1(x,z,t) = \varepsilon C S(x,z,t)) \tag{3}$$

e. g. the time dependent part of permittivity ε_1 is proportional to the sound field amplitude S and C is material constant of medium. From Maxwell´s equation (assuming $\boldsymbol{E}.grad\,\varepsilon(x,z,t) = 0$) we can obtain the wave equation in the form

$$\nabla^2 E(x,z,t) = \mu_0 \frac{\partial^2}{\partial t^2}\left[\varepsilon(x,z,t)E(x,z,t)\right] \tag{4}$$

Because the time variation of $\varepsilon\,(x,z,t)$ is much slower than that of E, we will only retain a term which does not contain the time differentiations of $\varepsilon\,(x,z,t)$:

$$\nabla^2 E(x,z,t) - \mu_0\varepsilon\frac{\partial^2 E(x,z,t))}{\partial t^2} = \mu_0\varepsilon_1(x,z,t)\frac{\partial^2 E(x,z,t))}{\partial t^2} \tag{5}$$

It is a wave equation often used to investigate strong interactions in acoustoptics. We will now introduce harmonic variations in the optical radiation and sound as complex functions in the forms

$$E(x,z,t)=\frac{1}{2}\sum_{m=-\infty}^{\infty}E_m(x,z)\exp[i(\omega+m\Omega)t]+c.c.\ ;\quad \frac{\varepsilon_1(x,z,t)}{\varepsilon}=\frac{1}{2}CS(x,z)\exp(i\Omega t)+c.c. \qquad (6)$$

where c.c. denotes complex conjugate. We assume the frequency mixing of fotons and acoustic phonons in the time-dependent phase of E according to equation (2). Substituing (6) to (5) and assuming $\Omega \ll \omega$ we obtain the infinite coupled-wave system

$$\nabla^2 E_m(x,z)+k^2E_m(x,z)+\frac{1}{2}k^2CS(x,z)E_{m-1}(x,z)+\frac{1}{2}k^2S^*(x,z)E_{m+1}(x,z)=0\,;\ k=\omega\sqrt{\mu_0\varepsilon} \qquad (7)$$

where k is propagation constant of electromagnetic (optical) radiation and the asterisk denotes complex conjugate. The quantity E_m is the complex amplitude of the mth order radiation wave function at frequency $\omega + m\Omega = \omega_m$.

We will now consider a uniform sound wave of the width of transducer L propagating along x axis:

$$S(x,z)=S(x)=S_0\exp(-iKx);\quad K=\frac{2\pi}{\Lambda} \qquad (8)$$

We expect that the spatial part of wave function E_m can be written as

$$E_m(x,z)=E_{0m}(x,z)\exp(-ikz\cos\Phi_m-ikx\sin\Phi_m) \qquad (9)$$

with the choice of angle Φ_m according to eq. (2).

Substituing (8), (9) and (2) into (7) we obtain

$$\frac{\partial^2E_{0m}}{\partial x^2}-2ik\sin\Phi_m\frac{\partial E_{0m}}{\partial x}-2ik\cos\Phi_m\frac{\partial E_{0m}}{\partial z}+\frac{1}{2}k^2CS_0^{*}E_{0m+1}\exp[-ikz(\cos\Phi_{m+1}-\cos\Phi_m)]+$$
$$+\frac{1}{2}k^2CS_0E_{0m-1}\exp[-ikz(\cos\Phi_{m-1}-\cos\Phi_m)]=0 \qquad (10)$$

Here we have assumed that within a wavelength λ the quantity E_{0m} and its first derivative does not change appreciably with z and hence the term $\partial^2E_{0m}/\partial z^2$ can be neglected. The last two terms in (10) represent the interactions between adjacent orders of scattered optical radiation with the sound waves. The first two terms on the left-hand-side are responsible for propagational diffraction and the third term represents the effect of the mth order of radiation travelling in direction slightly different from z. If the width L is not large the propagational diffraction can be neglected. Because the angles Φ_m are very small (if Φ_{inc} is very small or zero) we can assume that E_{0m} depends only from z. The equation (10) then becomes

$$2ik\cos\Phi_m\frac{\partial E_{0m}}{\partial z}=-\frac{1}{2}k^2CS_0^{*}E_{0m+1}\exp[-ikz(\cos\Phi_{m+1}-\cos\Phi_m)]-$$
$$-\frac{1}{2}k^2CS_0E_{0m-1}\exp[-ikz(\cos\Phi_{m-1}-\cos\Phi_m)] \qquad (11)$$

which has to be solved with the boundary condition (δ_{m0} is the Kronecker delta function):

$$E_{0m}=E_{0inc}\delta_{m0} \qquad (12)$$

The physical interpretation of (11) is that there exists a mutual coupling between neighboring orders of scattered radiation. But the phase of contributions varies with *z* and so the terms in exponential functions represent the lack of phase synchronism in this coupling process.

In Raman-Nath regime the interaction width *L* must be short. The degree of phase mismatch between different neighboring orders is therefore small and the exponential functions in (11) which represents this small phase asynchronism we can expand in a power series. Using Eq. (2) then we have

$$kz\left(\cos\Phi_{m\pm1} - \cos\Phi_m\right) \approx kz\left[\mp\frac{K}{k}\sin\Phi_{inc} \mp \left(m \pm \frac{1}{2}\right)\frac{K^2}{k^2} + \ldots\right] \tag{13}$$

For $\Phi_{inc} \cong 0$ the accumulated phase mismatch for order *m* at $z = L$ is negligible if

$$m\frac{K^2}{k}L \langle\langle 1 \tag{14}$$

This is the well-known criterion (for $m = 1$) $L \ll \Lambda^2/\lambda$ for an acoustoptic device operating in Raman – Nath regime. Using this criterion and assuming $\Phi_m \ll 1$, the Eq. (11), when for simplicity $S_0 = S_0^*$, we have

$$\frac{dE_{0m}(z)}{dz} = -\frac{ikCS_0}{4}\left[E_{0m-1}(z) - E_{0m+1}(z)\right] \tag{15}$$

This is a recursion relation for the Bessel functions $J_m(z)$. Hence for $z = L$

$$E_{0m}(L) = (-1)^m E_{0inc} J_m\left(\frac{kCS_0L}{2}\right) \tag{16}$$

3. Conclusion

Eq. (16) is very similar to the Raman-Nath solution [2]. The observable quantity, e. g. mean time value of the intensity of optical radiation is equal to $(1/2)\varepsilon_0 c E_{0m}^2 \sim J_m^2$. The presented model predicts in such a way the layout of intensities of multiple scattering maxima observable in standard acoustooptical device operating in Raman – Nath regime.

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References:

[1] F. J. Pedrotti, S. J. and L. S. Pedrotti: *Introduction to Optics,* Prentice Hall, Inc., New Yersey (U. S. A.) 1996.
[2] Raman, C. V. and Nath, N. S. N. (1935). *The diffraction of light by high frequency sound waves.* Proc. Indian Academy of Science, 2:406-420; 3:75-84; 3:119-125; 3:459-465, 4:222-242.
[3] Korpel A., *Acousto-optics*. In: Applied Solid State Science 3, Academic New York (1972)