BROWNIAN MOTION IN A MEDIUM WITH NONLINEAR FRICTION

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1. Introduction

According to the classical theory the motion of a Brownian particle (BP) with the mass m and radius R in a fluid is described by the Langevin equations (LEs) (d/dt is denoted by a dot)

$$\dot{x} = \upsilon, \quad m\dot{\upsilon} = -\gamma\upsilon + \sqrt{2D}\xi(t), \tag{1}$$

where $\upsilon(t)$ is the particle velocity and $x(t)$ is the particle position. The resistance force against the particle motion is the Stokes friction force proportional to the velocity, and the irregular impulses from the surrounding molecules are described by the (white) noise force ~ $\xi(t)$ with the properties $\langle \xi(t) \rangle = 0$ and $\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\delta(t-t')$. In Eq.(1), $D = k_B T\gamma$ is the noise intensity (k_B is the Boltzmann constant, T is the temperature and $\gamma = 6\pi R\eta$ the friction coefficient proportional to the dynamic viscosity η of the surrounding fluid). The effective diffusion coefficient of the particle, which also determines the intensity of the noise generated by Eq.(1), is given by the Einstein formula $D_{eff} = \gamma^{-2}D$.

Various generalizations of Eq.(1) during the past 20 years were used to describe the motion of living objects – molecular motors, flocks of animals, *etc*. The simplest case for the friction force of the individual BP is Rayleigh-Helmholtz friction, $\gamma(\upsilon) = \upsilon^2 - \upsilon_0^2$, which is interesting because of the fact that at $|\upsilon| < \upsilon_0$ the friction acts as an energy pump. Other models (not addressed here) that have attracted some attention were proposed by Schweitzer, Ebeling and Tilch [1], and by Schienbein and Gruler [2].

In this paper we will study the one-dimensional case when the friction coefficient γ and the noise intensity are functions of velocity. Our main focus is to derive the effective diffusion coefficient for the nonlinear Brownian motion (BM) with a general friction function. In this case the equation of motion for the BP should contain a stochastic (or "spurious") force proportional to $\partial_\upsilon D(\upsilon)$ [3]. Whereas in [4] the exact expression for the diffusion coefficient of the nonlinear BM when the LE does not include the spurious force was derived, here we take this force into account.

2. The diffusion coefficient for nonlinear Brownian motion

In the work [5], the main attention was given to the effective diffusion coefficient of the particle D_{eff}. This coefficient can be calculated from the mean square displacement of the particle as

$$D_{eff} = \lim_{t\to\infty} \frac{1}{2td} \sum_{j=1}^{d} \left\langle \left[x_j(t) - x_j(0) \right]^2 \right\rangle, \tag{2}$$

or using the Kubo relation for the velocity autocorrelation function,

$$D_{eff} = \frac{1}{d} \int_0^\infty \left\langle \upsilon(t)\upsilon(t+\tau) \right\rangle d\tau . \tag{3}$$

Here, d is the dimensionality of the system. The coefficient D_{eff} is important not only for the description of various kinds of the active and nonlinear BM, *e.g.* in physics and biology [6 - 8], but also for designing colored-noise generators [5]. Indeed, while the random force $\xi(t)$ describes the white noise with equal amplitudes of any frequency component of the force spectrum, equation (1) with nonlinear frictions, *e.g.* $\gamma(\upsilon) = \gamma\upsilon^{2\alpha}$, where α is a constant, can be regarded as a noise generator with very different properties. In such a generator, the diffusion coefficient D_{eff} corresponds to the noise intensity of the "velocity" υ. It is of particular interest to know the properties of this quantity and its dependence on the system parameters D, γ, and m.

Lindner´s theory [5] requires substantial improvements since for the nonlinear BM it incorporates the same random force as for the linear case. In the correct approach, when the friction force nonlinearly depends on the velocity υ, the random force should be changed: the intensity of the noise must also be the function of υ, $D(\upsilon)$, as an implication of the fluctuation-dissipation theorem. Due to this fact we have to add a "stochastic" or the so called spurious force, proportional to the derivative of $D(\upsilon)$ [3] in the LE. Equation (1) then becomes

$$m\dot{\upsilon} = -\gamma(\upsilon)\upsilon - am^{-1}\partial_\upsilon D(\upsilon) + \sqrt{2D(\upsilon)}\xi(t), \tag{4}$$

where the moments of random function $\xi(t)$ are defined as before. Equivalent to the LE (4) is the Fokker-Planck equation (FPE) for the distribution function $P(\upsilon,t)$

$$\partial_t P = m^{-2}\partial_\upsilon \left[D(\upsilon)\partial_\upsilon P \right] + m^{-1}\partial_\upsilon \left\{ \left[\gamma(\upsilon)\upsilon + m^{-1}(a+1/2)\partial_\upsilon D(\upsilon) \right] P \right\}. \tag{5}$$

The equilibrium solution of the FPE (5) must have the form of the Maxwell-Boltzmann distribution. After substituting this distribution into Eq.(5) for the state of equilibrium, the following fluctuation-dissipation relation for the case in question,

$$D(\upsilon) = k_B T \left[\gamma(\upsilon) + m^{-1}(a+1/2)\upsilon^{-1}\partial_\upsilon D(\upsilon) \right], \tag{6}$$

must reduce to the generalized Einstein relation $D(\upsilon) = k_B T\gamma(\upsilon)$, and hence the coefficient a in the equations of Langevin and Fokker-Planck must be $a = -1/2$.

Equation (5) can be established in different ways. Its most typical forms are: Ito´s [9, 10], Stratonovich´s [11, 12], and the kinetic´s form [3, 13, 14], corresponding to a = 1/2, 0, -1/2, respectively, and it is only in the last case that (6) coincides with the generalized Einstein relation [3].

In the kinetic interpretation, the FPE has the stationary solution $P_{st}(\upsilon)$ $= (m/2\pi k_B T)^{1/2} \exp(-m\upsilon^2/2k_B T)$. The normalization constant was determined from the normalization condition $\int_{\upsilon_{\min}}^{\upsilon_{\max}} P_{st}(\upsilon)d\upsilon = 1$ for the natural boundary conditions ($\upsilon_{\min} = -\infty$, $\upsilon_{\max} = +\infty$).

Now, we obtain the general formula for the effective diffusion coefficient D_{eff} for the one-variable case corresponding to the kinetic FPE (5). According to [8, 15], the diffusion coefficient is given by the simple formula

$$D_{eff} = \left\langle \Delta\upsilon^2 \right\rangle \tau_{corr}, \tag{7}$$

where $\tau_{corr} = \int_0^\infty \left[\left\langle \upsilon(t)\upsilon(t+\tau) - \left\langle \upsilon \right\rangle^2 \right\rangle \right] / \left\langle \Delta\upsilon^2 \right\rangle d\tau$ is the correlation time of the velocity and $\left\langle \Delta\upsilon^2 \right\rangle = \left\langle (\upsilon - \left\langle \upsilon \right\rangle)^2 \right\rangle$ is its variance. On the other side, in the work [16] we can find the analytical expression for the correlation time (S9.14). Using this fact, we can obtain the exact formula for the effective diffusion coefficient D_{eff} in the one-dimensional case ($d = 1$),

$$D_{eff} = \sqrt{\frac{mk_B T}{2\pi}} \int_{\upsilon_{\min}}^{\upsilon_{\max}} \gamma^{-1}(\upsilon) \exp\left[-U(\upsilon)\right] d\upsilon, \tag{8}$$

where $U(\upsilon) = m\upsilon^2 / 2k_B T$. If the general friction function $\gamma(\upsilon)$ is symmetric in υ, *e.g.* $\sim \upsilon^{2\alpha}$, and “well-behaved“, avoiding unphysical divergence of the velocity, in particular, the velocity can vary between two symmetric limiting values $\pm\upsilon_M$ [4] and the formula (8) can be replaced by

$$D_{eff} = 2k_B T \int_0^\infty \gamma^{-1}(\upsilon) P_{st}(\upsilon) d\upsilon. \tag{9}$$

The result (8) is valid for the kinetic representation of the FPE, which corresponds to the LE (4) with the spurious force. It is easily verified from (8) that the diffusion coefficient is inversely proportional to the friction coefficient γ. For normal BM ($\alpha = 0$) the Einstein formula $D_{eff} = \gamma^{-2} D$ takes place.

The diffusion coefficient in our nontrivial example, which is a nonlinear function $\gamma(\upsilon) = \gamma\upsilon^{2\alpha}$ discussed in [5], reads

$$D_{eff} = \frac{2^{-\alpha}(k_B T)^{1-\alpha} m^\alpha}{\sqrt{\pi}\gamma} \Gamma\left(\frac{1}{2} - \alpha\right), \tag{10}$$

if $2\alpha < 1$. This result coincides with the result from dimensional analysis. For $\alpha = -1$ from (10) one recovers the simple relation $D_{eff} = k_B^2 T^2 / m\gamma$.

3. Conclusion

In this contribution, we have studied the famous Langevin equation as a generator of the colored noise. We have described an example in which the Brownian particle produces a colored (correlated) noise. Instead of the Stokes friction, a force proportional to $\upsilon^{2\alpha}\upsilon$ is used; this choice is inspired by some actual models of Brownian motors [5]. The resulting Langevin equation is nonlinear and hardly solvable analytically. However, it was possible to obtain an exact expression for the effective diffusion coefficient.

In Section 2, we have derived an analytical expression for the diffusion coefficient of a one-dimensional Brownian motion with nonlinear friction, taking into account the spurious force in the Langevin equation. This coefficient is inversely proportional to the friction factor γ. We have shown that the expected Einstein formula takes place only in the case when the friction force $\gamma(\upsilon) \to \gamma$ (normal Brownian motion). This expression exactly corresponds to the theory. Expression (8) can be also used to explore other issues. Currently, our attempts are oriented in this direction.

At present, the theory of the nonlinear Brownian motion is intensively developed. We believe that along with the remarkable improvements of the experimental possibilities and broadening of the observable time and space scales it will find more and more applications, particularly in the physics of colloidal suspensions and the science and technology of electrical engineering.

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