BEAM ELEMENT WITH PIEZOELECTRIC LAYERS

***Vladimir Kutis*[1], *Justin Murin*[1], *Juraj Paulech*[1], *Juraj Hrabovsky*[1]**

[1] *Department of Applied Mechanics and Mechatronics, IEAE, FEEIT SUT Bratislava*
E-mail: vladimir.kutis@stuba.sk

Received 25 April 2012; accepted 27 April 2012.

1. Introduction

New materials have been introduced in the area of mechanisms and applications. Contemporary mechanical systems are focusing on minimizing size, active control and low energy consumption. All this attributes can be incorporated into term Micro Electro Mechanical Systems (MEMS). MEMS application usually contains multi layers structure and usually some of these layers have piezoelectric properties. Piezoelectric structures offer facilities to make motions. Piezoelectric layers can be also used to damping vibrations as an active damping or as an active sensor. For better understanding these problems we should start with constitutive equations of piezoelectric effect.

1.1 Piezoelectric constitutive equations

Piezoelectric constitutive equations describe the relation between mechanical and electrical quantities. The form of the constitutive equations depends on chosen mechanical and electrical quantities and can be expressed in two basic configurations. The first configuration is expressed by stress tensor components σ_{kl} and vector components of electric intensity E_k and has form

$$\begin{aligned} \varepsilon_{ij} &= d_{ijk} E_k + s_{ijkl}^E \sigma_{kl} \\ D_i &= \gamma_{ik}^{\sigma} E_k + d_{ikl} \sigma_{kl} \end{aligned} \tag{1}$$

where ε_{ij} are strain tensor components, D_i are components of electric displacement vector, d_{ijk} are tensor components of piezoelectric constants, γ_{ik}^{σ} are components of permittivity tensor on conditions constant mechanical stress and s_{ijkl}^E are components of compliance tensor on conditions constant electric intensity.

The constitutive equations can be also expressed by strain tensor components ε_{kl} and vector components of electric intensity E_k and has form

$$\begin{aligned} \sigma_{ij} &= c_{ijkl}^E \varepsilon_{kl} - e_{ijk} E_k \\ D_i &= e_{ijk} \varepsilon_{kl} + \gamma_{ik}^{\varepsilon} E_k \end{aligned} \tag{2}$$

where new quantities are components of stiffness tensor c_{ijkl}^E and components of piezoelectric modulus tensor e_{ijk}.

The other equations which play important role are relation between the components of strain tensor ε_{ij} and components of deformations u_i and relation between vector components of electric intensity E_i and electric potential φ. These relations can be expressed as

$$\varepsilon_{ij} = 1/2(u_{i,j} + u_{j,i})$$
$$E_i = -\varphi_{,i} \tag{3}$$

Tensor equations (1) and (2) can be expressed in matrix forms, which is more suitable for finite element formulations, using symmetry of mechanical and electrical quantities as well as material properties. Matrix formulations of constitutive equations (2) can be expressed as

$$\boldsymbol{\sigma} = \mathbf{c}^E \boldsymbol{\varepsilon} - \mathbf{e}^{\mathrm{T}} \mathbf{E}$$
$$\mathbf{D} = \mathbf{e}\boldsymbol{\varepsilon} + \boldsymbol{\gamma}^{\varepsilon} \mathbf{E} \tag{4}$$

where $\boldsymbol{\sigma}$ is stress vector with 6 components, $\mathbf{D}$ is electric displacement vector with 3 components, $\boldsymbol{\varepsilon}$ is strain vector with 6 components, $\mathbf{E}$ is vector of electric intensity with 3 components, $\mathbf{c}^E$ is 6x6 matrix of mechanical properties, $\boldsymbol{\gamma}^{\varepsilon}$ is 3x3 matrix of electrical properties and $\mathbf{e}$ is 3x6 matrix of piezoelectric properties.

1.2 Piezoelectric FEM equations

To obtain basic FEM equations for piezoelectric material, principle of virtual work is used. For static system, this principle for piezoelectric material can be expressed as

$$\int_V \delta\boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma} dV = \delta\mathbf{u}^T \mathbf{F}$$
$$-\int_V \delta\mathbf{E}^{\mathrm{T}} \mathbf{D} dV = \delta\boldsymbol{\varphi}^T \mathbf{Q} \tag{5}$$

where left side of (5) represents of virtual work of internal quantities and right side represents virtual work of external quantities. Using equations (5), relations (3), constitutive equations (4) and classical finite element procedure [1] we can obtain basic FEM equations of piezoelectric element in form

$$\begin{bmatrix} \mathbf{K}_{uu}^e & \mathbf{K}_{u\varphi}^e \\ \mathbf{K}_{\varphi u}^e & \mathbf{K}_{\varphi\varphi}^e \end{bmatrix} \begin{bmatrix} \mathbf{u}^e \\ \boldsymbol{\varphi}^e \end{bmatrix} = \begin{bmatrix} \mathbf{F}^e \\ \mathbf{Q}^e \end{bmatrix} \tag{6}$$

where $\mathbf{K}_{uu}^e$ is structural submatrix, $\mathbf{K}_{\varphi\varphi}^e$ is electric submatrix and $\mathbf{K}_{u\varphi}^e$ and $\mathbf{K}_{\varphi u}^e$ are coupling submatrix of element. $\mathbf{F}^e$ and $\mathbf{Q}^e$ are element nodal forces and charges, respectively. $\mathbf{u}^e$ and $\boldsymbol{\varphi}^e$ are element nodal displacements and potentials, respectively.

2. Piezoelectric beam finite element

Equations (6) can be also used for beam finite element with piezoelectric layers - see Fig.1. As we can see from Fig. 1, piezoelectric beam element has 6 structural degree of freedom ($u_i, v_i, \phi_i, u_j, v_j, \phi_j$) and 4 electrical degree of freedom ($\varphi_1, \varphi_2, \varphi_3, \varphi_4$). Structural loading is represented by 4 forces and 2 moments ($F_{xi}, F_{yi}, M_i, F_{xj}, F_{yj}, M_j$) and electrical loading by 4 charges (Q_1, Q_2, Q_3, Q_4). The length of beam is L, the width of beam element is b, the height of FGM core is h and height of piezolayer is h_p.

The core of the beam can be made of functionally graded materials (FGM), where material property - Young modulus $E(x, y)$ can be function of position x and y. Material of piezoelectric layers have constant Young modulus E_p, permitivity γ^{ε} and piezoelectric constants e_{21} and d_{21}.

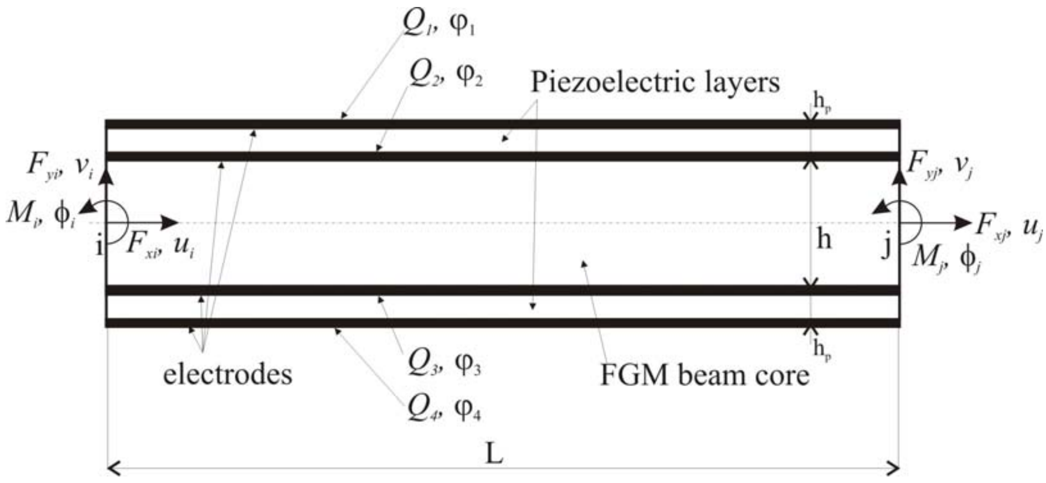

Fig.1: *Piezoelectric beam element with FGM core*

2.1 Equations for structural analysis

Structural submatrix $\mathbf{K}_{uu}^{e}$ for beam element with piezolayers can be expressed in form

$$\mathbf{K}_{uu}^{e} = \begin{bmatrix} k_u & 0 & -k_u & 0 & 0 & 0 \\ & k_{v1} & k_{v2} & 0 & -k_{v1} & k_{v3} \\ & & k_{v4} & 0 & -k_{v2} & k_{v5} \\ & S & & k_u & 0 & 0 \\ & & Y & & k_{v1} & -k_{v3} \\ & & & M & & k_{v6} \end{bmatrix} \tag{6}$$

where individual components contain the influence of FGM core stiffness and also the influence of piezolayers stiffness. The calculation of components is identical for classical multilayer or FGM beam without piezoelectric layer and is described in [2].

2.2 Equations for electric analysis

Electrical submatrix $\mathbf{K}_{\varphi\varphi}^{e}$ for beam element with piezolayers can be expressed in form

$$\mathbf{K}_{\varphi\varphi}^{e} = \frac{A_p L \gamma^{\varepsilon}}{h_p^2} \begin{bmatrix} -1 & 1 & 0 & 0 \\ S & -1 & 0 & 0 \\ & Y & -1 & 1 \\ & & M & -1 \end{bmatrix} \tag{6}$$

where A_p is cross-section of piezolayer and it is equal bh_p. All other parameters were defined in Fig.1.

2.3 Coupling of structural and electrical analysis

Submatrices $\mathbf{K}_{u\varphi}^{e}$ and $\mathbf{K}_{\varphi u}^{e}$, which describe coupling between mechanical and electrical quantities, can be expressed in following forms

$$\mathbf{K}_{u\varphi}^{e}=\frac{d_{21}E_p}{h_p}\begin{bmatrix}-A_p & A_p & -A_p & A_p\\ 0 & 0 & 0 & 0\\ A_y & -A_y & A_y & -A_y\\ A_p & -A_p & A_p & -A_p\\ 0 & 0 & 0 & 0\\ -A_y & A_y & -A_y & A_y\end{bmatrix};\ \mathbf{K}_{\varphi u}^{e}=\frac{e_{21}}{h_p}\begin{bmatrix}-A_p & 0 & A_y & A_p & 0 & -A_y\\ A_p & 0 & -A_y & -A_p & 0 & A_y\\ -A_p & 0 & A_y & A_p & 0 & -A_y\\ A_p & 0 & -A_y & -A_p & 0 & A_y\end{bmatrix} \quad (6)$$

where A_y is first moment of cross-section and for rectangular cross-section we can write $A_y = A_p/2\left(h+h_p\right)$.

3. Numerical experiments

The effectiveness of new piezoelectric beam element is shown on simple example, where piezoelectric beam (like in Fig.1) is fixed at left end and right end is free and is working like bending actuator. Beam core is made from FGM material. Upper and bottom parts (layers) of the beam are made of piezoelectric material PZT5A. Properties of FGM and 4 piezoelectric materials were published in [3]. Deformation of the beam, i.e. motion of the actuator, is caused by electric potential applied to electrodes of the piezoelectric layers.

Interior electrodes (see Fig.1) are coupled and grounded $\varphi_2 = \varphi_3 = 0V$. Required bending deformation is created by applied electric potential on electrodes with dependence on the polarization of the piezoelectric material. Applied electric potential is $\varphi_1 = \varphi_4 = 100V$. For this case we obtain following deformation parameters: $v_j = -1.853\times10^{-6}\,\mathrm{m}$, $\phi_j = 3.809\times10^{-5}\,\mathrm{rad}$. The same problem was calculated by code ANSYS [4], where 2D piezoelectric elements were used. Because plane elements do not have rotation as degree of freedom, we can compare only displacements. For this loading case, we obtained $v_j^{ANSYS} = -1.853\times10^{-6}\,\mathrm{m}$. As we can see, there is very good correspondence in our new piezoelectric beam results and ANSYS 2D elements results. With application of negative potential on inside electrodes $\varphi_1 = \varphi_4 = -100V$ we get also reversed deformation with same absolute values of results.

4. Conclusion

The paper presents new beam finite element with piezoelectric layers, where core of the beam can be made of FGM materials. Such combinations of materials is very attractive for machatronic applications, because material composition of FGM core can be optimized for design stress state and deformation can be controled by voltages on electrodes. The beam finite element can be used for analysis of such systems very effectively and accurately.

Acknowledgement

This work was financially supported by grant of Science and Technology Assistance Agency no. APVV-0450-10 and by Grant Agency KEGA, grant No. 015STU-4/2012 and VEGA 1/0534/12.

References:

[1] D. Burnett: Finite Element Analysis, Addison-Wesley, Massachusetts, USA, (1987).
[2] V. Kutiš, J. Murín, R. Belák, J. Paulech: *Computers & Structures*, **89**, 1192 (2011).
[3] V. Kutiš: Habilitation Thesis, STU Bratislava (2010).
[4] ANSYS, Theory manual (2011).