MODAL ANALYSES OF FUNCTIONALLY GRADED MATERIAL BEAMS

Justín Murín[1], Mehdi Aminbaghai[2], Juraj Hrabovský[1], Vladimír Kutiš[1], Juraj Paulech[1]

[1] *Department of Applied Mechanics and Mechatronics IEAE FEI STU in Bratislava*
[2]*Vienna University of Technology, Institute for Mechanics of Materials and Structures*
Karlsplatz 13, A-1040 Vienna, Austria

justin.murin@stuba.sk

Received 06 April 2012; accepted 12 April 2012.

1. Introduction

Functionally graded materials (FGMs) are made as a mixture of two or more different constituents which are usually of the same dimensions and geometry. The variation of macroscopic material properties can be caused by varying the volume fraction of the constituents or by varying constituent's material properties (caused by non-homogeneous temperature field, for example). The FGM links, beams, shells and solids are built as the mechanical parts of the mechanical, mechatronic or electronic devices in practical applications (from micro to macro scales) like the sensors and actuators, electric current conductors, etc. The multhiphysical (electro-thermo-structural) and dynamical analyses have to be done in virtual prototyping of such parts and devices. The semi-analytical and numerical methods (predominantly the Finite Element Method - FEM) have been used in solution of the main field equations. Several homogenizations' methods have been used in modeling of the FGM structures like the extended or improved mixture rules [1], [2], the Representative Volume Element method – RVE and other multiscale methods [3]. The homogenized effective material properties have been considered in the main equations derivation.

In this contribution, the differential equation of the homogenized FGM beam deflection and its solution will be presented which induce the transfer relations of the 2nd order beam theory. They will be used in the modal analysis of the FGM beams with polynomial continuous longitudinal and transversal variation of material properties. The direct integration method [4] and multilayer method [5] can be used in homogenization of the spatial varying material properties. The FGM beams are considered to be resting on longitudinal variable (Winkler) elastic foundation. The shear correction function has been derived from which the average shear correction factor has been calculated. Numerical experiments were performed to calculate the eigenfrequencies and corresponding eigenmodes of chosen FGM beams

2. Homogenization of the spatial varying material properties

Let us consider a two nodal straight beam element with predominantly rectangular cross-sectional area *A* and quadratic moment of inertia *I* (Fig. 1). The following approach can be used also for other cross-sectional area types. Both the fibers volume fraction $v_f(x,y)$ and the matrix volume fraction $v_m(x,y)$ are chosen as a polynomial functions of *x*, and with continuous and symmetrical variation through its height *h* with respect to the neutral plane of the beam. The volume fractions are assumed to be constant through the cross-section depth *b*. At each point of the beam it holds: $v_f(x,y)+v_m(x,y)=1$. The values of the volume fractions at the nodal points are denoted by indices *i* and *j*. The assumption of polynomial variation enables an easier establishing of the beam equations and allows modeling many common continuous variations of beam parameters.

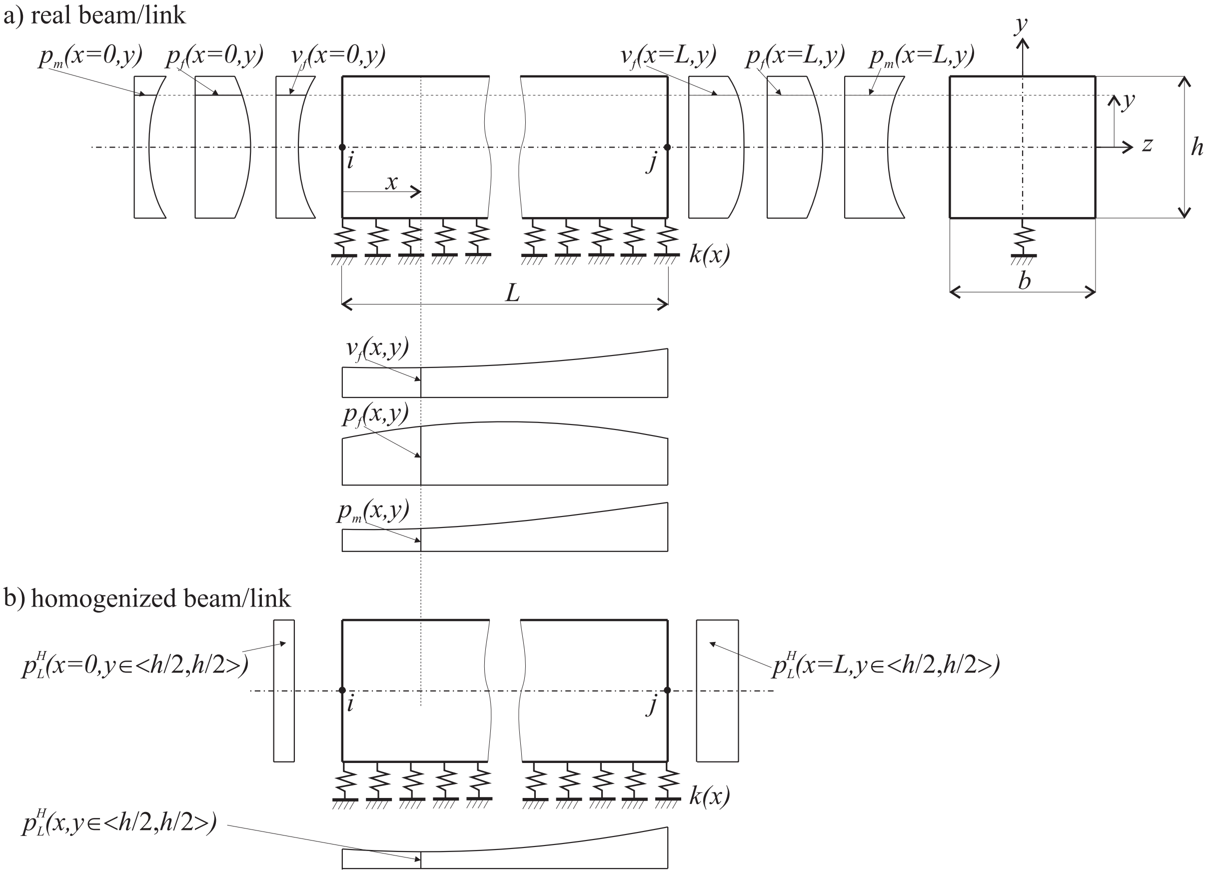

Fig.1: *FGM beam with spatial variation of material properties.*

The material properties of the constituents (fibres - $p_f(x,y)$ and matrix - $p_m(x,y)$) vary analogically as stated by the variation of the volume fractions. For effective material property $p(x,y)$ in the real beam we have got:

$$p(x,y) = v_f(x,y)p_f(x,y) + v_m(x,y)p_m(x,y). \tag{1}$$

In our case the elasticity modulus $E(x,y)$, Poisson ratio $\nu(x,y)$, and mass density $\rho(x,y)$ have been calculated by expression (1). The FGM shear modulus can be calculated by expression:

$$G(x,y) = \frac{E(x,y)}{2(1+\nu(x,y))}. \tag{2}$$

In the homogenization of the spatial varying material properties (1), (2) the direct integration method will be used [4]. From the assumption that the respective property (e.g. stiffness) of the real beam must be equal to the analogical property of the homogenized beam, the homogenized longitudinal elasticity modules for: tension – compression $E_L^{NH}(x)$, bending $E_L^{MH}(x)$, shear $G_L^H(x)$, and the homogenized mass density $\rho_L^H(x)$, can be calculated [6], respectively. The homogenized material properties have been used in establishing the shear correction function and the differential equations of free vibration of the homogenized FGM beam.

3. Differential equations of the FGM beam free vibration rested on Winkler foundation

According to [7], the main coupled equations of the 2nd order beam theory (including the inertia forces, shear force and axial force) are:

$$R' = -q + kw - \mu\omega^2 w \qquad\qquad M' = Q + m + \overline{\mu}\omega^2\varphi \tag{3}$$

$$\varphi' = -\frac{M}{EI} \Rightarrow M = -EI\varphi' - EI\kappa^e \qquad\qquad w' = \varphi + \frac{Q}{G\tilde{A}} \Rightarrow Q = G\tilde{A}w' - G\tilde{A}\varphi \tag{4}$$

Here, q is the distributed transversal load (see Fig. 2); m is the distributed bending moment; κ^e is he applied beam curvature; k is the modulus of elastic Winkler foundation; μ is the mass distribution; $\overline{\mu}$ is the mass inertial moment distribution; ω is the natural eigenfrequency; R is the transversal force; Q is the shear force; M is the bending moment; φ is the angle of cross-section rotation; w is the beam bending; EI is the bending stiffness and $G\tilde{A}$ is the reduced shear stiffness of the homogenized FGM beam. We assume that all the above quantities are the polynomial functions of x.

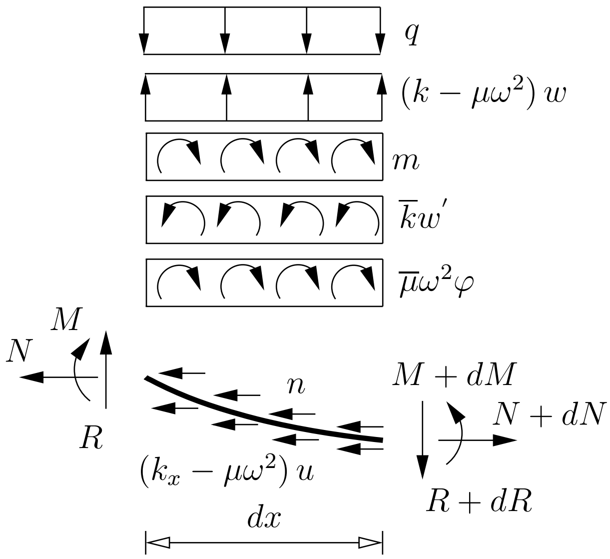

Fig. 2: *The force equilibrium in the deformed element configuration.*

The relation between the transversal and shear force is:

$$Q = -(\overline{k} + N^{II})w' - N^{II}\psi + R \tag{5}$$

where $N^{II} \equiv N$ is the resultant axial force of the 2nd order beam theory, ψ is the beam rotation imperfection, and $\overline{k}$ is the elastic foundation modulus for beam rotation. The derivation of the four coupled differential equations and their solution for the buckling force and eigenfrequency will be described in [6] in detail.

4. Numerical experiment

Cantilever beam (Fig. 3) is made of a mixture of titanium carbide TiC (fibres) – the elasticity modulus $E_f = 480.0\,\mathrm{GPa}$, the mass density $\rho_f = 4920\,\mathrm{kgm^{-3}}$, the Poisson's ratio $\nu_f = 0.20$; and aluminum Al 6061-TO (matrix) – the elasticity modulus $E_m = 69.0\,\mathrm{GPa}$, the mass density $\rho_m = 2700\,\mathrm{kgm^{-3}}$, the Poisson's ratio $\nu_m = 0.33$; [11]. Its geometry is given with: $b = h = 0.01\mathrm{m}$, $L = 0.1\mathrm{m}$.

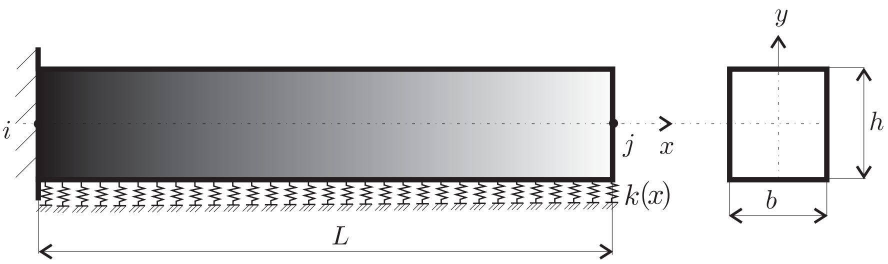

Fig. 3**:** *Cantilever beam with planar variation of material properties.*

Variation of the fibres volume fraction has been chosen as the polynomial function:

$$v(x,y) = \frac{400000000x^3y^2}{3} - \frac{400x^3}{3} - 20000000x^2y^2 + 200x^2 + 40000y^2,$$

that is drawn in Fig. 4.

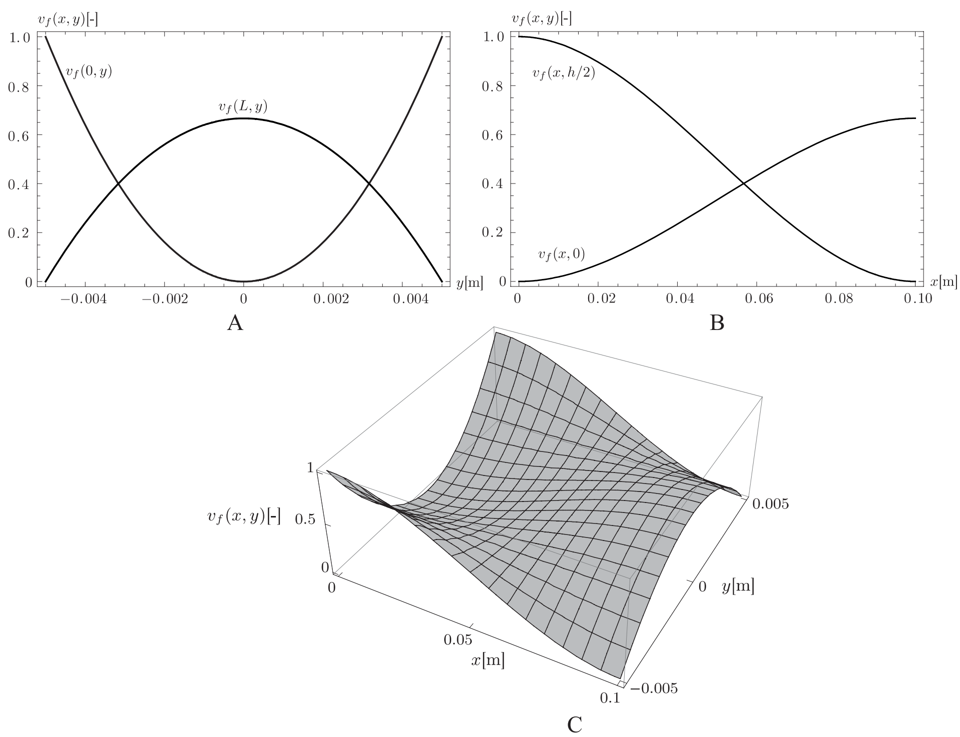

Figure 4: *Volume fraction variation: A – transversal variation at point i and j,*
B – longitudinal variation at the top and bottom of the beam,
C – planar variation along the beam length and beam height

Using the extended mixture rule (1), (2), a spatial distribution of the effective elasticity modulus $E(x,y)$ in [GPa], the Poisson's ratio $\nu(x,y)$ [-], shear modulus $G(x,y)$ in [GPa] and mass density $\rho(x,y)$ in [kgm^{-3}] have been calculated. The effective beam properties have been calculated using the expressions derived in [6]. Strong Winkler elastic foundation modulus has been chosen as a varying non-linear function of *x*: $k(x) = 5000 - 1000x + 6000x^2$ [kN/m^2]. The first three bending eigenfrequencies have been found using the differential equations (3) – (4) for 3 options of the shear force deformation effect consideration: without the shear effect - $\varsigma(x) = 0$; with the average shear correction factor - $k^{sm} = 0.75$; and with the shear correction function - $k^s(x)$. The same problem has been solved using a very fine mesh – 12000 of 2D PLANE42 elements of the FEM program ANSYS [8]. The average relative difference Δ[%] (for three chosen values of N force of each) between eigenfrequencies calculated by our method and the ANSYS solution has been evaluated. To show the effect of normal force N on the eigenfrequency, its positive value (tension), negative value (compression), and equal to zero have been taken into account. The nonzero values of the axial force have been chosen by $N \cong \pm 0.75 N_{Ki}^{II}$ in all cases calculated, where N_{Ki}^{II} is the 2nd order beam theory buckling force. The buckling force has been calculated from the differential equations by setting $\omega = 0$. Table 1 shows the 1st bending eigenfrequency:

Tab. 1: *The 1st eigenfrequency.*

N [kN] Option:	52	0	-52	Δ [%]	N_{Ki}^{II} [kN]
$\varsigma(x) = 0$	2 048.9	1 599.2	808.4	1.35	-67.71
$k^{sm} = 0.75$	2 031.5	1 585.8	797.0	0.30	-67.24
$k^{s}(x)$	2022.3	1 582.8	797.7	0.11	-67.31
ANSYS	2 019.5	1 582.6	796.2		

The eigenmodes for all the considered options have been calculated which will be presented in [6] in detail. Significant influence of the shear force and axial force on the eigenforms has been observed.

5. Conclusion

The solution results in the Table 1 obtained by our very effective method show very good agreement with the ANSYS solution (with very fine mesh). The best agreement of both results is for the 1st eigenfrequency; the shear force effect is meaningful in all calculated cases; the most accurate results have been obtained by consideration the shear correction function $k^{s}(x)$ in the eigenfrequency calculation. The eigenfrequency can be significantly influenced by the axial force.

Acknowledgement

This work was financially supported by grant VEGA-1/0534/12, KEGA 015STU-4/2012, and APVV-0676-11.

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[8] ANSYS Swanson Analysis System, Inc., 201 Johnson Road, Houston. PA 15342/1300, USA