OSCILLATOR DRIVEN INTERNALLY BY CORRELATED NOISE

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1. Introduction

One hundred years have passed since the explanation of the phenomenon of Brownian motion (BM) by Einstein, Smoluchowski and Langevin. Like other great inventions, a need was generated for a special mathematical description. It has taken the new concept of stochastic differential equations to express the molecular kinetic aspect of this phenomenon. The fluctuation force exerted on the Brownian particle (BP) by the molecules of a surrounding medium was depicted by an additive noise in the differential equation of motion of a BP [1]. For small (micron-sized or smaller) particles, the always present thermal fluctuations essentially influence their motion. For a BP in a potential well we can then speak about a noisy oscillator. Experimentally such situations have been realized for colloidal particles in optical traps [2]. In several papers the experiments on a BP confined in a moving harmonic well have been theoretically described [3, 4]. However, this description can be applied only for long observation times since it ignores the inertial and memory effects in the particle motion. For a colloidal particle in a solvent these effects should necessarily be taken into account at short times when the expected dynamics is ballistic. Also at long times the mean square displacement (MSD) of the particle can exhibit an "anomalous" (different from that in the Einstein theory) time dependence. In the present contribution we describe the BM with memory, using the generalized Langevin equation (LE). The application of the Vladimirsky rule [5] allowed us to exactly solve this integro-differential equation when its memory kernel exponentially decreases with the time.

2. Oscillator driven by correlated noise

If the random force driving the particles is not the delta-correlated white noise but a coloured noise, the resistance force against the particle motion cannot be arbitrary (e.g., it cannot be the Stokes one as in the standard Einstein and Langevin theories) but must obey the fluctuation-dissipation theorem. Then the equation of motion for the BP has a non-Markovian form of the generalized LE [6] that, for a particle of mass M in a harmonic potential $U = kx^2/2$, is

$$M\dot{\upsilon}(t) + M\int_0^t \Gamma(t-t')\upsilon(t')\mathrm{d}t' + M\omega^2 x(t) = f(t), \tag{1}$$

where $\langle f(t)f(0)\rangle = k_B T\Gamma(t)$. The memory in the system is described by the kernel $\Gamma(t) = \omega_M \omega_m \exp(-\omega_m t)$. Here, $\omega = (k/M)^{1/2}$ is the oscillator frequency and $\upsilon(t) = \dot{x}(t)$ is the velocity of the BP. Let the correlated random force $f(t)$ arises from the standard LE $m\dot{u}(t) + \gamma u(t) = \eta(t)$ with the white noise $\eta(t)$ and the friction factor γ in the Stokes force proportional to the velocity $u(t)$ of the surrounding particles. The characteristic relaxation times of the particles of mass m and the BP of mass M, respectively, are $\tau_m = 1/\omega_m = m/\gamma$ and $\tau_M = 1/\omega_M = M/\gamma$. According to the rule first derived in [5], the stochastic LE can be converted to a deterministic equation for the quantity $V(t) = \dot{\xi}(t)$, where $\xi(t)$ is the particle MSD [7], $V(0) = \dot{\xi}(0) = 0$, and the force $f(t)$ is replaced with $2k_B T$. Using the Laplace transformation, this equation for $\tilde{V}(s) = \Lambda\{V(t)\}$ reads

$$\tilde{V}(s) = \frac{2k_B T}{M}\frac{s+\omega_m}{s^3 + \omega_m s^2 + \left(\omega_m\omega_M + \omega^2\right)s + \omega_m\omega^2}. \tag{2}$$

The inverse transformation, found after expanding this expression in simple fractions, is

$$V(t) = \frac{2k_B T}{M}\sum_{i=1}^{3} A_i \exp(s_i t),$$

and the MSD is obtained by simple integration,

$$\xi(t) = \int_0^t V(t')\mathrm{d}t' = \frac{2k_B T}{M}\sum_{i=1}^{3}\frac{A_i}{s_i}\left[\exp(s_i t) - 1\right]. \tag{3}$$

Here s_i are the roots of the cubic polynomial in the denominator of Eq.(2) and $A_1 = (s_1 + \omega_m)(s_1 - s_2)^{-1}(s_1 - s_3)^{-1}$. The coefficients A_2 and A_3 have the same form with the cyclic change of the indexes $1\to2\to3\to1$. These constants obey the following relations:

$$\sum_i \frac{A_i}{s_i} = \frac{\omega_m}{s_1 s_2 s_3} = -\frac{1}{\omega^2}, \quad \sum_i A_i = 0, \quad \sum_i A_i s_i = 1, \quad \sum_i A_i s_i^2 = 0, \quad \sum_i A_i s_i^3 = -\left(\omega_m \omega_M + \omega^2\right),$$

which can be used in calculations of the asymptotic behaviour of the solution (3). For $\xi(t)$ at $t \to 0$ we find (the main term being independent on the driving force intensity)

$$\xi(t) \approx \frac{k_B T}{M} t^2 \left(1 - \frac{\omega_m \omega_M + \omega^2}{12} t^2 + \ldots\right). \tag{4}$$

At long times the asymptote can be written in the form (independent on m), different from the Einstein result for diffusion,

$$\xi(t) \approx \frac{2k_B T}{M\omega^2}\left[1 - \exp\left(-\frac{\omega^2 t}{\omega_M}\right)\right]. \tag{5}$$

In the absence of the harmonic force ($\omega \to 0$) we have the expected result $\xi(t \to \infty) \approx 2k_B Tt / \gamma$, which follows also from the exact solution (3) and the properties of A_i.

Now, let us take into account the possibility that the harmonic well moves with the velocity υ^* along the axis x [3]. The position of the BP will be denoted by $x_t = x + x^*$, where x obeys the stochastic LE (1) and x^* is the solution of the inhomogeneous deterministic equation

$$\ddot{x}^* + \int_0^t \Gamma(t-t')\dot{x}^*(t')\mathrm{d}t' + \omega^2 x^* = \omega^2 \upsilon^* t. \tag{6}$$

The full solution obeys the GLE (1) with x in the last term on the left hand side replaced by x_t - $\upsilon^* t$ and υ changed to $\upsilon_t = \dot{x}_t$. The solution for $x^*(t)$ with the initial conditions $x^*(0) = \dot{x}^*(0) = 0$ can be easily obtained in a similar way as above. Using the Laplace transformation we obtain $x^*(t) = \omega^2 \upsilon^* \sum_i A_i s_i^{-1}\left\{[\exp(s_i t) - 1]s_i^{-1} - t\right\}$, with the following limits at short and long times, respectively: $x^*(t \to 0) \approx \omega^2 \upsilon^* t^3 / 6$ and $x^*(t \to \infty) \approx \upsilon^* t$. The full MSD of the particle is calculated as $X(t) = \xi(t) + [x^*(t)]^2$.

3. Conclusion

The theory of the Brownian motion is intensively developed and along with the remarkable improvements of the experimental possibilities it finds more and more applications, especially in the science and technology of small systems. It has been found that in many situations [6] the standard Langevin equation should be generalized to take into account the effects of finite correlation time in the noise driving the particles. In our contribution, a specific problem of the motion of a Brownian particle under the influence of an exponentially correlated stochastic force has been solved within the classical consideration. As distinct from the previous works, the inertial effects have been taken into account. We have examined the case of a free particle as well as the motion of a particle in a moving harmonic trap (a stochastic oscillator with memory). From the generalized Langevin equation the exact MSD $\xi(t)$ of the particle has been calculated and analyzed in detail. At short times $\xi(t)$ corresponds to the ballistic motion $\sim t^2$. At $t \to \infty$ the MSD converges to a constant strongly depending on the oscillator frequency ω and agrees with the Einstein diffusion limit when $\omega \to 0$. The full MSD for a trapped particle corresponds to the experiments on colloids [3]. It can be also used to describe the charge fluctuations in nanoscale electric circuits in contact with the thermal bath [8]. The solution of the latter however requires a further development of the presented approach to the case when $f(t)$ in Eq.(1) is the quantum noise.

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